Pseudospin symmetry solution of the Dirac equation with an angle-dependent potential



2008 J. Phys. A: Math. Theor. 41 045302

(http://iopscience.iop.org/1751-8121/41/4/045302)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 045302 (11pp) doi:10.1088/1751-8113/41/4/045302

# Pseudospin symmetry solution of the Dirac equation with an angle-dependent potential

**Cüneyt Berkdemir**[1] **and Ramazan Sever**[2]

[1] Department of Physics, Erciyes University, Kayseri, 38039, Turkey
[2] Department of Physics, Middle East Technical University, Ankara, 06531, Turkey

E-mail: berkdemir@erciyes.edu.tr and sever@metu.edu.tr

Received 16 October 2007, in final form 6 December 2007
Published 15 January 2008
Online at stacks.iop.org/JPhysA/41/045302

**Abstract**

The pseudospin symmetry solution of the Dirac equation for spin 1/2 particles moving within the Kratzer potential connected with an angle-dependent potential is investigated systematically. The Nikiforov–Uvarov method is used to solve the Dirac equation. All of the studies are performed for the exact pseudospin symmetry ($SU2$) case and also the exact spin symmetry case is given briefly in the appendix. Bound-state solutions are presented to discuss the contribution of the angle-dependent potential to the relativistic energy spectra in the full description case which is either unavailable or excessively complicated.

PACS numbers: 03.65.Ge, 03.65.Pm, 02.30.Gp, 31.30.Jv

## 1. Introduction

The relativistic descriptions introduced to understand the relativistic behaviour of spin 1/2 particles show that the solution of the Dirac equation with mixed potentials for particles such as atoms, nuclei and hadrons play a central role in a realistic nuclear system [1]. The electron or muon in muonic atoms subject to the relevant potentials exhibit Coulombic behaviour and move independently in the relativistic potentials, involving mixtures of attractive scalar and repulsive vector potentials with opposite signs. A wide interest is to study the pseudospin symmetry considered as a relativistic symmetry in nuclear physics to rename the single-particle levels in the shell model [2, 3]. The idea of pseudospin symmetry was introduced to explain the quasidegeneracy in some nuclei between single-nucleon states with quantum numbers $(n, \ell, j = 1/2)$ and $(n-1, \ell+2, j = \ell+3/2)$, where $n$, $\ell$ and $j$ are the radial, the orbital and the total angular momentum quantum numbers, respectively. These levels have the same '$pseudo$' orbital angular momentum quantum number, $\tilde{\ell} = \ell + 1$, and pseudospin quantum number, $\tilde{s} = 1/2$. For example, for $\left(ns_{\frac{1}{2}}, (n-1)d_{\frac{3}{2}}\right)$ one has $\tilde{\ell} = 1$, for $\left(np_{\frac{3}{2}}, (n-1)f_{\frac{5}{2}}\right)$ one

has $\tilde{\ell} = 2$, etc. Pseudospin symmetry is exact when doublets with $j = \tilde{\ell} \pm \tilde{s}$ are degenerate [4–8].

Recently, analytic solutions of the Dirac equation with spin and pseudospin symmetry have received considerable attention. Ginocchio [9] solved the Dirac equation for the triaxial, axial and spherical harmonic oscillators with spin symmetry. Chen *et al* [10] solved the Dirac equation for two kinds of harmonic oscillator potentials with exact spin symmetry, $\Delta = V - S = 0$, and pseudospin symmetry, $\Sigma = V + S = 0$, by using a Dirac Hamiltonian with scalar $S$ and vector $V$ potentials quadratic in space coordinates. Moreover, these authors discussed the origin of pseudospin symmetry and its breaking in real nuclei in the relativistic mean field theory. Guo *et al* [11] studied the s-wave Dirac equation for the Woods–Saxon potential with spin and pseudospin symmetry and found a constant relationship between $V$ and $S$ potentials, i.e., $V - S =$ constant and $V + S =$ constant. In the general sense, pseudospin symmetry occurs for $V + S =$ constant in the Dirac equation. Very recently, Zou *et al* [12] have investigated the Dirac equation for the Eckart potential with an equally mixed case. The authors have only considered the case of the spin symmetry limitation, i.e., imposing the difference between $V$ and $S$ potentials to zero. Hence, it is of considerable interest to study the Dirac–Eckart problem with general spin symmetry and pseudospin symmetry. Jia *et al* [13] investigated the analytic solutions of the Dirac equation for the Eckart potential with spin and pseudospin symmetry in terms of the supersymmetric quantum mechanics approach and function analysis method. Qiang *et al* [14] presented an application of the exact quantization rule to the relativistic solution of the rotational Morse potential with pseudospin symmetry. As an inspiration of this study, Berkdemir [15] has found a special solution of the pseudospin symmetry in the relativistic Morse potential by using the Nikiforov–Uvarov (NU) method [16]. This alternative method gives a systematical framework to obtain a spacial solution of the corresponding second-order differential equations, i.e., Schrödinger, Dirac and Klein–Gordon equations for various potentials [17, 18]. The general purpose of the NU method is to solve the hypergeometric-type second-order linear differential equations in terms of special orthogonal functions [19]. For a given potential, the associated differential equation is reduced to a spacial-type equation with an appropriate coordinate transformation. This equation can be written in the following form:

$$\psi''(s) + \frac{\tilde{\tau}(s)}{\sigma(s)}\psi'(s) + \frac{\tilde{\sigma}(s)}{\sigma^2(s)}\psi(s) = 0, \tag{1}$$

where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials, at most second-degree, and $\tilde{\tau}(s)$ is a first-degree polynomial.

In this paper, our aim is to present an analytical solution for the Dirac–Kratzer problem with pseudospin symmetry including an angle-dependent term. We introduce the 'sum' and the 'difference' potentials defined as $\Sigma = V + S$ and $\Delta = V - S$, respectively. When $\Sigma = 0$ or $\Delta = 0$, the Dirac Hamiltonian is invariant under a $SU(2)$ symmetry [20]. This is a general feature independent of the particular forms of $S$ and $V$. We start dealing with the case $\Sigma = 0$ ($S = -V$) and also using the Kratzer potential plus an angle-dependent potential for the case $\Delta \neq 0$, i.e., $\Delta \equiv \Delta(r, \theta; A, B, C) = -A/r + (B + C\cot^2\theta)/r^2$ [21, 22]. We also give a short summary of the relationship between pseudospin and spin symmetry solutions for the cases $\Delta = 0$ ($S = V$) and also $\Sigma \neq 0$ at the end of this paper. Moreover, we follow the basic solution procedure of the NU method given in [23]. To keep away from notational complications and to preserve the well-defined discussions, we will follow the latest study of Guo *et al* [24].

## 2. Pseudospin symmetry solution for the relevant potentials

### *2.1. Separating variables of the Dirac equation with pseudospin symmetry*

Relativistic symmetry is valid if the potentials are triaxial and spherical [7]. In the case of spherical coordinates, the Dirac equation of a single-nucleon with mass $\mu$ is given by

$$[\vec{\alpha}\cdot\vec{p}+\beta(\mu+S(\vec{r}))+V(\vec{r})]\psi(\vec{r})=E\psi(\vec{r}), \tag{2}$$

where the unites $\hbar$ and $c$ are set to 1. The term $V(\vec{r})$ comes from the time component of a four-vector potential. The space components are usually called the vector potential and have been put to zero here. The $S(\vec{r})$ piece is introduced as an extra space dependent mass term, not obtained by minimal coupling. $E$ is the relativistic energy eigenvalues of the Dirac particle, $\vec{p}$ is the three-momentum operator, $\vec{\alpha}$ and $\beta$ are $4\times 4$ Dirac matrices given in the following forms, respectively,

$$\vec{\alpha}=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\qquad \beta=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}. \tag{3}$$

Here $\vec{\sigma}$ is a three-vector whose components are two-dimensional Pauli matrices and $I$ stands for the $2\times 2$ identity matrix. For spherical nuclei, the nucleon angular momentum $\vec{J}$ and $\hat{K}=-\beta(\vec{\sigma}\cdot\vec{L}+1)$ commute with the Dirac Hamiltonian $H$, independent of the spin and pseudospin symmetries, where $\vec{L}$ is the orbital angular momentum. According to the complete set of the conserved quantities $(H,\hat{K},\vec{J}^2,J_z)$, the Dirac eigenfunctions are labelled by the conserved quantum numbers $j$ and $\kappa$ rather than orbital angular momentum or pseudo-orbital angular momentum. The eigenfunctions have the form

$$\psi(\vec{r})=\begin{pmatrix}f(\vec{r})\\ g(\vec{r})\end{pmatrix} \tag{4}$$

where $f(\vec{r})$ and $g(\vec{r})$ are the upper (large) and lower (small) spinor components of the wave function $\psi(\vec{r})$, respectively. The spherically symmetric Dirac eigenfunction can then be written according to its upper and lower components. The Dirac equation given in equation (1) may be reduced to a set of two-coupled ordinary differential equations in terms of these components, namely

$$\vec{\sigma}\cdot\vec{p}\,f(\vec{r})=(E+\mu-\Delta)g(\vec{r}), \tag{5}$$

$$\vec{\sigma}\cdot\vec{p}\,g(\vec{r})=(E-\mu-\Sigma)f(\vec{r}), \tag{6}$$

where the 'difference' and 'sum' potentials are defined by $\Delta=V(\vec{r})-S(\vec{r})$ and $\Sigma=V(\vec{r})+S(\vec{r})$, respectively. To study the exact pseudospin symmetry, $\Sigma$ is equalized to zero, i.e., $\Sigma=0$. In this case, equation (6) is simplified into the following form:

$$f(\vec{r})=\frac{\vec{\sigma}\cdot\vec{p}}{E-\mu}g(\vec{r}). \tag{7}$$

Substituting equation (7) into equation (5), a Schrödinger-like equation is obtained for the lower spinor component of the Dirac equation,

$$[\vec{p}^2+(E-\mu)\Delta]g(\vec{r})=(E^2-\mu^2)g(\vec{r}). \tag{8}$$

When $\Delta$ is taken to be the Kratzer potential plus an angle-dependent potential, i.e., $\Delta\equiv\Delta(r,\theta;A,B,C)=-A/r+(B+C\cot^2\theta)/r^2$, equation (8) becomes

$$\left[-\nabla^2+(E-\mu)\left(-\frac{A}{r}+\frac{B}{r^2}+\frac{C\cos^2\theta}{r^2\sin^2\theta}\right)\right]g(\vec{r})=(E^2-\mu^2)g(\vec{r}). \tag{9}$$

To see the decoupling of pseudospin and pseudo-orbital angular momentum, the lower spinor component $g(\vec{r})$ can be split into two parts as a spin up or a spin down, i.e., $\binom{1}{0}$ or $\binom{0}{1}$, multiplied by a function of the spherical coordinates. In this case, $g(\vec{r})$ can be written in spherical coordinates as follows:

$$g(\vec{r}) \equiv g(r,\theta,\varphi) = \frac{G(r)}{r}\Theta(\theta)\Phi(\varphi)\widetilde{\chi}_m, \tag{10}$$

where $m$ is the projection of angular momentum on the $z$-axis and its value is $\pm 1/2$. $\widetilde{\chi}_m$ are two-component spinors named spin up or spin down. Substituting equation (10) into equation (9), the angular and radial wave functions are separated as

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2} + \widetilde{\Lambda}^2\Phi(\varphi) = 0, \tag{11}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta} + \left(\nu - C(E-\mu)\frac{\cos^2\theta}{\sin^2\theta} - \frac{\widetilde{\Lambda}^2}{\sin^2\theta}\right)\Theta(\theta) = 0, \tag{12}$$

$$\frac{\mathrm{d}^2G(r)}{\mathrm{d}r^2} + \left[(E^2-\mu^2) - (E-\mu)\left(\frac{B}{r^2} - \frac{A}{r}\right) - \frac{\nu}{r^2}\right]G(r) = 0, \tag{13}$$

where $\widetilde{\Lambda}^2$ and $\nu$ are separation constants and $\nu$ represents the $\kappa(\kappa-1)$ (or $\tilde{\ell}(\tilde{\ell}+1)$) in terms of the spin–orbit coupling term $\kappa$. For the bound state, we have to consider the boundary conditions $\Phi(\varphi+2\pi) = \Phi(\varphi)$ in equation (11). Furthermore, $\Theta(0)$ and $\Theta(\pi)$ have a finite value in equation (12) and the radial function boundary conditions are given as $G(0) = 0$ and $G(\infty) = 0$ in equation (13). Considering these boundary conditions, the solution of equation (11) can be obtained immediately as

$$\Phi_{\widetilde{\Lambda}}(\varphi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{\mathrm{i}\widetilde{\Lambda}\varphi}, \qquad \widetilde{\Lambda} = 0, \pm 1, \pm 2, \ldots. \tag{14}$$

Here, $\widetilde{\Lambda}$ is a quantum number coming from the $\varphi$-dependent part of the wave function and the projection of the pseudo-orbital angular momentum along the symmetry axis [20].

### 2.2. *Solution of the $\theta$-dependent equation*

In order to obtain the solution of equation (12), a new variable $x = \cos\theta$ is introduced. By setting

$$\widetilde{C} = C(E-\mu) + \nu = \ell'(\ell'+1), \tag{15}$$

where $\ell'$ is a new pseudo-orbital angular momentum which comes from the contribution of the angle-dependent potential, and

$$\widetilde{D} = \nu - \widetilde{\Lambda}^2, \tag{16}$$

equation (12) becomes

$$\frac{\mathrm{d}^2\Theta(x)}{\mathrm{d}x^2} - \frac{2x}{1-x^2}\frac{\mathrm{d}\Theta(x)}{\mathrm{d}x} + \frac{1}{(1-x^2)^2}(\widetilde{D} - \widetilde{C}x^2)\Theta(x) = 0. \tag{17}$$

The comparison of equations given in equation (1) and equation (17) gives us the following polynomials;

$$\widetilde{\tau}(x) = -2x, \qquad \sigma(x) = 1-x^2, \qquad \widetilde{\sigma}(x) = \widetilde{D} - \widetilde{C}x^2. \tag{18}$$

The basic concepts of the NU method given in [23] are followed to solve equation (17). According to [23], the polynomial $\pi(x)$ is obtained by inserting the above equations into equation (11) of [23]:

$$\pi(x) = \pm\sqrt{x^2(\widetilde{C} - k) + k - \widetilde{D}}, \tag{19}$$

where $k$ is a constant. According to the NU method to find polynomial solutions of equation (17), the expression under the square root must be the square of a linear polynomial in $x$. Therefore, the polynomial $\pi(x)$ is found in the following possible values:

$$\pi(x) = \pm\begin{cases} \sqrt{\widetilde{C} - \widetilde{D}} & \text{for} \quad k_+ = \widetilde{C} \\ x\sqrt{\widetilde{C} - \widetilde{D}} & \text{for} \quad k_- = \widetilde{D}, \end{cases} \tag{20}$$

where $k_+$ and $k_-$ are the roots of $k$. In order to obtain a physical solution valid in the NU method, the derivation of polynomial $\tau(x) = \widetilde{\tau}(x) + 2\pi(x)$ must have a negative value. Thus, we take

$$k_- = \widetilde{D}, \tag{21}$$

so that

$$\pi(x) = -x\sqrt{\widetilde{C} - \widetilde{D}}. \tag{22}$$

Then, $\tau(x)$ is arranged as follows,

$$\tau(x) = -2x(1 + \sqrt{\widetilde{C} - \widetilde{D}}), \tag{23}$$

and hence $\tau'(x)$ becomes

$$\tau' = -2(1 + \sqrt{\widetilde{C} - \widetilde{D}}). \tag{24}$$

Using equation (8) of [23], $\lambda_{n'}$ and $\lambda$ are obtained as

$$\lambda_{n'} = n'^2 + n' + 2n'\sqrt{\widetilde{C} - \widetilde{D}}, \tag{25}$$

$$\lambda = k_- + \pi'(x) = \widetilde{D} - \sqrt{\widetilde{C} - \widetilde{D}}. \tag{26}$$

Comparing equation (25) with equation (26) and recalling the value of $\widetilde{C}$ given in equation (15), the new pseudo-orbital angular momentum is obtained as follows:

$$\begin{aligned} &\ell'(\ell' + 1) - [\widetilde{\Lambda}^2 + C(E - \mu)] - \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)} = n'^2 + n' + 2n'\sqrt{\widetilde{\Lambda}^2 + C(E - \mu)}, \\ &\ell'(\ell' + 1) = \left(n' + \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)}\right)\left(n' + \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)} + 1\right), \\ &\ell' = n' + \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)}. \end{aligned} \tag{27}$$

The term $\ell'$ in equation (27) can be named the 'modified' pseudo-orbital angular momentum, since the contribution which comes from the angle-dependent potential damages the usual pseudo-orbital angular momentum $\widetilde{\ell}$. The parameter $\ell'$ does not need to be an integer. However, the difference between the parameter $\ell'$ and the square root term in equation (27) has to be an integer:

$$n' = \ell' - \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)}, \qquad n' = 0, 1, 2, \ldots, \tag{28}$$

where $n'$ is a quantum number coming from the $\theta$-dependent part of the wave function and corresponds to the number of quanta for oscillations along the symmetry axis [20]. Moreover, there is a relationship between $\tilde{\ell}(\tilde{\ell} + 1)$ and $\ell'(\ell' + 1)$ as follows:

$$\tilde{\ell}(\tilde{\ell} + 1) = \ell'(\ell' + 1) - C(E - \mu). \tag{29}$$

If the contribution parameter $C$ is fixed to zero, the last term on the right-hand side of equation (29) will have disappeared. Hence the 'modified' pseudo-orbital angular momentum will be reduced to the usual pseudo-orbital angular momentum.

**Table 1.** Analytical results and parameters of the radial equation.

| Parameters | Analytical results |
|---|---|
| $\widetilde{\tau}(r)$ | 0 |
| $\sigma(r)$ | $r$ |
| $\widetilde{\sigma}(r)$ | $-\varepsilon^2 r^2 + \delta r - \gamma$ |
| $k_{\pm}$ | $\delta \pm \varepsilon\sqrt{1+4\gamma}$ |
| $\pi(r)$ | $\frac{1}{2} \pm \frac{1}{2}(2\varepsilon r + \sqrt{1+4\gamma})$ for $k_+ = \delta + \varepsilon\sqrt{1+4\gamma}$ |
| | $\frac{1}{2} \pm \frac{1}{2}(2\varepsilon r - \sqrt{1+4\gamma})$ for $k_- = \delta - \varepsilon\sqrt{1+4\gamma}$ |
| $\tau(r)$ | $-2\varepsilon r + 1 - \sqrt{1+4\gamma}$ |
| $\tau'$ | $-2\varepsilon(\varepsilon > 0)$ |
| $\lambda$ | $\delta - \varepsilon(1 + \sqrt{1+4\gamma})$ |
| $\lambda_n$ | $2n\varepsilon$ |

### *2.3. Solution of the radial equation*

We will now deal with the solution of the radial equation in equation (13). The acceptable bound-state solutions are only possible if $|E| < \mu$. Letting

$$-\varepsilon^2 = (E^2 - \mu^2), \quad (\varepsilon > 0) \qquad \delta = (E-\mu)A, \qquad \gamma = \nu + (E-\mu)B, \tag{30}$$

and also substituting these abbreviations into equation (13), we obtain a radial equation as follows:

$$\frac{\mathrm{d}^2 G(r)}{\mathrm{d}r^2} + \frac{1}{r^2}(-\varepsilon^2 r^2 + \delta r - \gamma)G(r) = 0. \tag{31}$$

To solve this equation, we again use the NU method, its basic concepts briefly given in [23]. To keep away from the repetition of the solution steps of the NU method, we summarized the solution steps in table 1 (recalling equations (8) and (11) of [23]). According to the solution procedure of the NU method, there are four possible solutions for $\pi(r)$. These solutions can be valid mathematically, but in the physical nature the eigenvalue solutions can be obtained from one of the four possible solutions for $\pi(r)$. To obtain the energy eigenvalues for the pseudospin–orbit dependent solution of the Dirac particle within the Kratzer potential plus an angle-dependent potential, it is enough to equate $\lambda$ with $\lambda_n$ given at the end of table 1, where $n$ is the radial quantum number coming from the radial part of the wave function. Thus, we easily write a comprehensive solution including the angle-dependent contributions,

$$\varepsilon^2(1 + 2n + \sqrt{1+4\gamma})^2 - \delta^2 = 0, \tag{32}$$

$$(\mu^2 - E^2)(1 + 2n + \sqrt{1 + 4[\ell'(\ell'+1) + (E-\mu)(B-C)]})^2 - (E-\mu)^2 A^2 = 0, \tag{33}$$

where $\gamma$ is recalled from equation (30), using the equality of $\nu = \ell'(\ell'+1) - C(E-\mu)$ from equation (15). From equation (33) it is very complicated to find relativistic energy eigenvalues $E$, keeping in mind that $\ell'$ is energy dependent. To discuss the restrictions on the relativistic energy spectra, a full description must be prepared in terms of the parameters of the relevant potential. From equations (27) and (33), it can be found that the relativistic energy eigenvalues $E$ depend on $n, n'$ and $\widetilde{\Lambda}$ as well as the parameters $A, B, C$ and $\mu$ ($E = E(n, n', \widetilde{\Lambda}; A, B, C, \mu)$). Moreover, the degenerate energy spectrum is also correlated with these parameters. However, bound-state solutions can be obtained by setting the parameters in the relevant potential to appropriate values.

## 3. Particular cases

### *3.1. The $B = 0$ and $C = 0$ case*

In this subsection, we set to zero two parameters of the relevant potential. Then, the $\Delta$ potential with $B = 0$ and $C = 0$ leads to the well-known Coulomb potential $\Delta = -A/r$. We should address the constant $A = Z\alpha > 0$ with the fine structure constant $\alpha = 1/137$, which is correct only in three dimensions. The corresponding parameters become

$$\ell' = n' + |\widetilde{\Lambda}| = \widetilde{\ell} = 0, 1, 2 \ldots, \qquad \gamma = \widetilde{\ell}(\widetilde{\ell} + 1) = \kappa(\kappa - 1), \tag{34}$$

and the relativistic energy spectrum is simplified into the following form,

$$(\mu^2 - E^2)(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2 - (E - \mu)^2 A^2 = 0. \tag{35}$$

To see the restrictions on the relativistic energy spectrum, equation (35) can be written as follows:

$$(E + \mu) = -\frac{(E - \mu)A^2}{(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2}, \tag{36}$$

$$E = -\mu\left(\frac{(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2 - A^2}{(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2 + A^2}\right). \tag{37}$$

Considering equation (37), there are two types of analysis on the relativistic energy spectrum. The first one is negative-energy bound states which are available for the condition of $(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2 > A^2$, when $\Delta$ is a Coulomb potential and $\Sigma = 0$. The second one represents positive-energy bound states, i.e., $(1 + 2n + \sqrt{1 + 4\kappa(\kappa - 1)})^2 < A^2$. In this condition, $E$ is bigger than zero because the numerator is negative and it is actually for anti-particles subject to a strong Coulomb field. Moreover, there could not be any solutions of equation (37) for particles, only for anti-particles. This can be seen from the fact that if the parameter $A$ of the $\Delta = -A/r$ is increased slowly form zero, the bound states will emerge from $E = -\mu$. The conditions of $\Delta = 0$ and $\Sigma = 0$ are related by charge conjugation or chiral transformations, as remarked by Castro *et al* [25].

As a comment, we can recall the relativistic energy spectrum coming from the spin symmetry solution. This solution is briefly given in the appendix. For the $B = 0$ and $C = 0$ cases, the general solution of the spin symmetry given in equation (A.15) is simplified as follows:

$$E = \mu\left(\frac{(1 + 2n + \sqrt{1 + 4\kappa(\kappa + 1)})^2 - A^2}{(1 + 2n + \sqrt{1 + 4\kappa(\kappa + 1)})^2 + A^2}\right). \tag{38}$$

Equation (38) is different from equation (37) from the point of view of the relativistic energy spectrum. For the condition $(1+2n+\sqrt{1 + 4\kappa(\kappa + 1)})^2 > A^2$, the energy spectrum is positive in equation (38). This is *not* equal to the well-known positive energy spectrum of the relativistic Dirac–Coulomb problem [26]. However, the correct nonrelativistic limit can be achieved in the case of weak coupling, i.e., $Z\alpha \ll 1$ and $|E - \mu| \ll \mu$.

### *3.2. The $B = 0$ case*

When $B = 0$ is imposed, the relevant potential reduces to the Coulomb potential plus an angle-dependent potential. The corresponding parameters become

$$\ell' = n' + \sqrt{\widetilde{\Lambda}^2 + C(E - \mu)}, \qquad \gamma = \ell'(\ell' + 1) - C(E - \mu), \tag{39}$$

and the relativistic energy spectrum is reduced to the following form,

$$E = -\mu\left(\frac{(1+2n+\sqrt{1+4[\ell'(\ell'+1)-C(E-\mu)]})^2 - A^2}{(1+2n+\sqrt{1+4[\ell'(\ell'+1)-C(E-\mu)]})^2 + A^2}\right). \tag{40}$$

The above equation can be solved analytically to obtain the relativistic energy spectrum, keeping in mind that $\ell'$ is energy dependent. One of the solution procedures is to follow a graphical method for solving this transcendental equation (see [25] for details). To mention the negative-energy bound states, the expression in the numerator of equation (40) must be positive. However, this is not adequate for the negative states. Considering $\widetilde{\Lambda}^2 \geqslant 0$, in order to insure $\ell'$ real, $C \leqslant 0$ must be satisfied due to the condition $E - \mu < 0$. Moreover, for the $C > 0$ case, there are two conditions, i.e., $\widetilde{\Lambda}^2 \geqslant C|E-\mu|$ and $1 + 4\ell'(\ell'+1) \geqslant 4C|E-\mu|$. The same analysis can be also prepared for the spin symmetry solution given in equation (A.15).

### *3.3. The $C = 0$ case*

When $C = 0$ is imposed, the relevant potential turns into the well-known Kratzer potential, i.e., $\Delta = -A/r + B/r^2$. The corresponding parameters become

$$\ell' = n' + |\widetilde{\Lambda}| = \widetilde{\ell} = 0, 1, 2\ldots, \qquad \gamma = \widetilde{\ell}(\widetilde{\ell}+1) + B(E-\mu), \tag{41}$$

and the relativistic energy spectrum is reduced to the following form,

$$E = -\mu\left(\frac{(1+2n+\sqrt{1+4[\widetilde{\ell}(\widetilde{\ell}+1)+B(E-\mu)]})^2 - A^2}{(1+2n+\sqrt{1+4[\widetilde{\ell}(\widetilde{\ell}+1)+B(E-\mu)]})^2 + A^2}\right). \tag{42}$$

Equation (42) is solvable to discuss the relativistic energy spectrum for the Kratzer potential (see [25] for details). For the negative-energy bound-state solutions, the expression in the numerator of equation (42) must be positive. Furthermore, the expression within the square roots must be real. To satisfy this condition, the value of the parameter $B$ should be arranged as a positive or negative one. For the positive value of $B$, the condition of $1+4\widetilde{\ell}(\widetilde{\ell}+1) \geqslant 4B|E-\mu|$ must be satisfied within the square roots. The other value of $B$ ($B < 0$) does not cause any restriction within the square roots. However, the well-known Kratzer potential is valid for the positive value of the parameter $B$.

## 4. Remarks and conclusion

We have studied the analytic solution to understand some key qualitative features of angle-dependent potentials for the Dirac equation with spin and pseudospin symmetry. The results we obtained in this work can be used in nuclear physics to investigate a family of the Kratzer potential under the conditions of the pseudospin and also spin symmetry views. In the contents of the paper, we have solved exactly the Dirac equation for the Kratzer potential plus an angle-dependent potential by using the NU method. The negative or positive bound-state solutions are discussed for several particular cases, such as the Coulomb potential, Kratzer potential and their angle-dependent forms. Moreover, the solution of the Dirac equation with spin symmetry is also studied to compare with the pseudospin symmetry cases. The effect of the contribution constant $C$ which comes from the angle-dependent potential is also discussed by setting its value to appropriate interval.

**Acknowledgments**

The authors wish to thank the referees for their helpful comments and suggestions. This research was partially supported by the Scientific and Technological Research Council of Turkey.

## Appendix

The Dirac equation given in equation (4) can be reduced to a set of two-coupled equations:

$$\vec{\sigma}\cdot\vec{p}\,f(\vec{r}) = (E+\mu-\Delta)g(\vec{r}), \tag{A.1}$$

$$\vec{\sigma}\cdot\vec{p}\,g(\vec{r}) = (E-\mu-\Sigma)f(\vec{r}). \tag{A.2}$$

In the limit of the exact spin symmetry, $\Delta$ must be set to zero. Hence, equation (A.1) takes

$$g(\vec{r}) = \frac{\vec{\sigma}\cdot\vec{p}}{E+\mu}f(\vec{r}) \qquad (E+\mu\neq 0). \tag{A.3}$$

Substituting equation (A.3) into equation (A.2), the upper spinor component of the Dirac equation becomes

$$[\vec{p}^{\,2}+(E+\mu)\Sigma]f(\vec{r}) = (E^2-\mu^2)f(\vec{r}). \tag{A.4}$$

When $\Sigma$ is taken as the Kratzer potential plus an angle-dependent potential, equation (A.4) becomes

$$\left[-\nabla^2+(E+\mu)\left(-\frac{A}{r}+\frac{B}{r^2}+\frac{C\cos^2\theta}{r^2\sin^2\theta}\right)\right]f(\vec{r}) = (E^2-\mu^2)f(\vec{r}). \tag{A.5}$$

To separate the differential equation in equation (A.5) for the angular and radial parts, $f(\vec{r})$ is written as

$$f(\vec{r})\equiv f(r,\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\frac{F(r)}{r}\Theta(\theta)\,\mathrm{e}^{\mathrm{i}\widetilde{\Lambda}\varphi}\widetilde{\chi}_m, \qquad \widetilde{\Lambda}=0,\pm1,\pm2,\ldots \tag{A.6}$$

where $m$ is $\pm1/2$ and $\widetilde{\chi}_m$ is a two-component spinor. Substituting equation (A.6) into equation (A.5), the angular and radial parts become

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}+\left(\widetilde{\nu}-C(E+\mu)\frac{\cos^2\theta}{\sin^2\theta}-\frac{\widetilde{\Lambda}^2}{\sin^2\theta}\right)\Theta(\theta)=0, \tag{A.7}$$

$$\frac{\mathrm{d}^2F(r)}{\mathrm{d}r^2}+\left[(E^2-\mu^2)-(E+\mu)\left(\frac{B}{r^2}-\frac{A}{r}\right)-\frac{\widetilde{\nu}}{r^2}\right]F(r)=0, \tag{A.8}$$

where $\widetilde{\nu}$ represents the $\kappa(\kappa+1)$ (or $\ell(\ell+1)$). $\ell$ is the usual orbital angular momentum number. For the bound states, $\Theta(0)$ and $\Theta(\pi)$ have a finite value in equation (A.7) and the radial function boundary conditions are given as $F(0)=0$ and $F(\infty)=0$ in equation (A.8). In order to derive the solution of equation (A.7), a new variable $x=\cos\theta$ is introduced. By setting

$$\widetilde{C} = C(E+\mu)+\widetilde{\nu} = \ell'(\ell'+1), \tag{A.9}$$

where $\ell'$ is a new pseudo-orbital angular momentum which comes from the contribution of the angle-dependent potential and

$$\widetilde{D} = \widetilde{\nu}-\widetilde{\Lambda}^2, \tag{A.10}$$

equation (A.7) becomes

$$\frac{d^2\Theta(x)}{dx^2} - \frac{2x}{1-x^2}\frac{d\Theta(x)}{dx} + \frac{1}{(1-x^2)^2}(\widetilde{D} - \widetilde{C}x^2)\Theta(x) = 0. \tag{A.11}$$

When equation (A.11) is compared with equation (17), it can be clearly seen that both equations are the same. By following the solution procedure given in section 2.2, we can obtain the 'modified' pseudo-orbital angular momentum for the spin symmetry case,

$$\ell' = n' + \sqrt{\widetilde{\Lambda}^2 + C(E+\mu)}. \tag{A.12}$$

Equation (A.12) is different from equation (27) from the point of view of the sign within the square root.

Let us now define

$$-\varepsilon^2 = (E^2 - \mu^2), \quad (\varepsilon > 0) \qquad \delta = (E+\mu)A, \qquad \gamma = \widetilde{\nu} + (E+\mu)B, \tag{A.13}$$

and substitute into equation (A.8). The radial equation becomes

$$\frac{d^2F(r)}{dr^2} + \frac{1}{r^2}(-\varepsilon^2 r^2 + \delta r - \gamma)F(r) = 0. \tag{A.14}$$

We solve equation (A.14) by following the solution procedure given in section 2.3. By using the analytical results listed in table 1, we easily write a comprehensive solution including the angle-dependent contributions for the spin symmetry case:

$$(\mu^2 - E^2)(1 + 2n + \sqrt{1 + 4[\ell'(\ell'+1) + (E+\mu)(B-C)]})^2 - (E+\mu)^2A^2 = 0. \tag{A.15}$$

Although equation (A.15) has an analytical solution for certain values of the relevant potential parameters, the expression of $E$ is too complicated to solve it and to find a valid energy spectrum for the Kratzer potential related to an angle-dependent potential.

## References


[1] Leviatan A 2004 *Phys. Rev. Lett.* **92** 202501
Leviatan A 2005 *Int. J. Mod. Phys.* E **14** 111
[2] Arima A, Harvey M and Shimizu K 1969 *Phys. Lett.* B **30** 517
[3] Hecht K T and Alder A 1969 *Nucl. Phys.* A **137** 129
[4] Sugawara-Tanabe K, Meng J, Yamaji S and Arima A 1999 *J. Phys. G: Nucl. Part. Phys.* **25** 811
[5] Lisboa R, Malheiro M, de Castro A S, Alberto P and Fiolhais M 2004 *Phys. Rev.* C **69** 024319
[6] Ginocchio J N and Madland D G 1998 *Phys. Rev.* C **57** 1167
[7] Ginocchio J N and Leviatan A 2001 *Phys. Rev. Lett.* **87** 072502
[8] Ginocchio J N 2002 *Phys. Rev.* C **66** 064312
[9] Ginocchio J N 1997 *Phys. Rev. Lett.* **78** 436
Ginocchio J N 2004 *Phys. Rev.* C **69** 034318
Ginocchio J N 2005 *Phys. Rep.* **414** 165
[10] Chen T-S, Lü H-F, Meng J, Zhang S-Q and Zhou S-G 2003 *Chin. Phys. Lett.* **20** 358
[11] Guo J-Y, Fang X Z and Xu F X 2005 *Nucl. Phys.* A **757** 411
Guo J-Y and Sheng Z-Q 2005 *Phys. Lett.* A **338** 90
Guo J-Y, Xu Q and Han J C 2004 *J. At. Mol. Phys. (in China)* **21** 679
[12] Zou X, Yi L Z and Jia C S 2005 *Phys. Lett.* A **346** 54
[13] Jia C-S, Wang X-G, Yao X-K, Chen P-C and Xian W 1998 *J. Phys. A: Math. Gen.* **31** 4763
Jia C-S, Wang J-Y, He S and Sun L-T 2000 *J. Phys. A: Math. Gen.* **33** 6993
Jia C-S, Guo P and Peng X-L 2006 *J. Phys. A: Math. Gen.* **39** 7737
Jia C-S, Liu J-Y, He L and Sun L-T 2007 *Phys. Scr.* **75** 388
[14] Qiang W-C, Zhou R-S and Gao Y 2007 *J. Phys. A: Math. Theor.* **40** 1677
[15] Berkdemir C 2006 *Nucl. Phys.* A **770** 32
[16] Nikiforov A F and Uvarov V B 1988 *Special Functions of Mathematical Physics* (Basel: Birkhauser)
[17] Flügge S 1971 *Practical Quantum Mechanics I and II* (Berlin: Springer)

[18] Berkdemir C and Han J 2005 *Chem. Phys. Lett.* **409** 203
Berkdemir C, Berkdemir A and Sever R 2005 *Phys. Rev.* C **72** 027001
Berkdemir A, Berkdemir C and Sever R 2006 *Mod. Phys. Lett.* A **21** 2087
Berkdemir C, Berkdemir A and Han J 2006 *Chem. Phys. Lett.* **426** 329
Berkdemir C 2007 *Am. J. Phys.* **75** 81
[19] Szego G 1959 *Orthogonal Polynomials* Revised edition (Providence, RI: American Mathematical Society)
[20] Ginocchio J N, Leviatan A, Meng J and Zhou S-G 2004 *Phys. Rev.* C **69** 034303
[21] Kratzer A 1920 *Z. Phys.* **3** 289
[22] Dong S-H, Chen C-Y and Lozada-Cassou M 2005 *Int. J. Quantum Chem.* **105** 453
[23] Berkdemir C, Berkdemir A and Sever R 2006 *J. Phys. A: Math. Gen.* **39** 13455
[24] Guo J-Y, Han J-C and Wang R-D 2006 *Phys. Lett.* A **353** 378
[25] de Castro A S, Alberto P, Lisboa R and Malherio M 2006 *Phys. Rev.* C **73** 054309
[26] Wong M K F and Yeh H Y 1982 *Phys. Rev.* D **25** 3396
Su J Y 1985 *Phys. Rev.* A **32** 3251
Goodman B and Ignjatović S R 1997 *Am. J. Phys.* **65** 214
Dong S-H 2003 *J. Phys. A: Math. Gen.* **36** 4977
Alhaidari A D 2004 *Ann. Phys., NY* **312** 144
Alhaidari A D 2004 *J. Phys. A: Math. Gen.* **37** 11229
Chen C-Y 2005 *Phys. Lett.* A **339** 283
Dutra A de S and Hott M 2006 *Phys. Lett.* A **356** 215
Alhaidari A D, Bahlouli H and Al-Hasan A 2006 *Phys. Lett.* A **349** 87
Kulikov D A and Tutik R S 2006 *11th Int. Conf. on Mathematical Methods in Electromagnetic Theory (Kharkiv, Ukraine, June 26–29)*